Reducing the Number of Axioms Required to Define a Kleene Algebra

K Prahlad Narasimhan

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A *Kleene Algebra* is defined to be an algebraic structure: $\mathcal{K} = \langle K, +, \cdot, ^*, 0, 1\rangle$ satisfying the following axioms[1]:

$$a + (b + c) = (a + b) + c \tag{1}$$

$$a + b = b + a \tag{2}$$

$$a + 0 = a \tag{3}$$

$$a + a = a \tag{4}$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \tag{5}$$

$$a \cdot 1 = 1 \cdot a = a \tag{6}$$

$$a \cdot 0 = 0 \cdot a = 0 \tag{7}$$

$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{8}$$

$$(a + b) \cdot c = a \cdot c + b \cdot c \tag{9}$$

$$1 + a \cdot a^* = a^* \tag{10}$$

$$1 + a^* \cdot a = a^* \tag{11}$$

$$b + a \cdot c \leq c \implies a^* \cdot b \leq c \tag{12}$$

$$b + c \cdot a \leq c \implies b \cdot a^* \leq c \tag{13}$$

where $\leq$ refers to the natural partial ordering in $\mathcal{K}$: $a \leq b \iff a + b = b$.

[1] *"Automata and Computability"* by D.C Kozen

While multiplying two elements a and b of the algebra, $a \cdot b$ is denoted by ab for convenience.

Lemma. Axiom 11 is redundant.

Proof. We need to prove that we can derive the equation $1 + a^*a = a^*$ from the other axioms. Let $b = 1 + a^*a$. We have,

$$
\begin{aligned}
& 1 + aa^* = a^* \\
\implies & a + aa^*a = a^*a \\
\implies & 1 + a(1 + a^*a) = 1 + a^*a \\
\implies & 1 + ab = b \\
\implies & 1 + ab \leq b \\
\implies & a^* \leq b && \text{(Axiom 12)}
\end{aligned}
$$

Also, we have,

$$
\begin{aligned}
& 1 \leq a^* \\
\implies & a \leq aa^* \leq a^* \\
\implies & a \leq a^* \\
\implies & a + aa^* \leq a^* + aa^* \\
\implies & a + aa^* \leq a^* + (1 + aa^*) \\
\implies & a + aa^* \leq a^* + a^* && \text{(Axiom 10)} \\
\implies & a + aa^* \leq a^* && \text{(Axiom 4)} \\
\implies & a^*a \leq a^* && \text{(Axiom 12)} \\
\implies & 1 + a^*a \leq 1 + a^* \\
\implies & 1 + a^*a \leq a^* \\
\implies & b \leq a^*
\end{aligned}
$$

This proves our supposition since $\leq$ is a partial ordering. $\square$